VOLUME 79 SEPARATE No. 183

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

APRIL, 1953

GROUP LOADINGS APPLIED TO THE ANALYSIS OF FRAMES

By I. F. Morrison

STRUCTURAL DIVISION

Copyright 1953 by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.

PRICE $0.50 PER COPY

GUIDEPOST FOR TECHNICAL READERS

"Proceedings-Separates" of value or significance to readers in various fields are here listed, for convenience, in terms of the Society's Technical Divisions. Where there seems to be an overlapping of interest between Divisions, the same Separate number may appear under more than one item. For a description of papers open to discussion refer to the current issue of *Civil Engineering*.

Technical Division	*Proceedings-Separate Number*
Air Transport	108, 121, 130, 148, 163, 172, 173, 174, 181, 187 (Discussion: D-75, D-93, D-101, D-102, D-103, D-108, D-121)
City Planning	151, 152, 154, 164, 167, 171, 172, 174, 177 (Discussion: D-65, D-86, D-93, D-99, D-101, D-105, D-108, D-115, D-117)
Construction	160, 161, 162, 164, 165, 166, 167, 168, 181, 183, 184, (Discussion: D-75, D-92, D-101, D-102, D-109, D-113, D-115, D-121, D-126, D-128, D-136)
Engineering Mechanics	145, 157, 158, 160, 161, 162, 169, 177, 179, 183, 185, 186, (Discussion: D-24, D-33, D-34, D-49, D-54, D-61, D-96, D-100, D-122, D-125, D-126, D-127, D-128, D-135, D-136)
Highway	144, 147, 148, 150, 152, 155, 163, 164, 166, 168, 185 (Discussion: D-103, D-105, D-108, D-109, D-113, D-115, D-117, D-121, D-123, D-128)
Hydraulics	154, 159, 164, 169, 175, 178, 180, 181, 184, 186, 187 (Discussion: D-90, D-91, D-92, D-96, D-102, D-113, D-115, D-122, D-123, D-135)
Irrigation and Drainage	148, 153, 154, 156, 159, 160, 161, 162, 164, 169, 175, 178, 180, 184, 186, 187 (Discussion: D-102, D-109, D-117, D-135)
Power	130, 133, 134, 135, 139, 141, 142, 143, 146, 148, 153, 154, 159, 160, 161, 162, 164, 169, 175, 178, 180, 184, 186 (Discussion: D-96, D-102, D-109, D-112, D-117, D-135)
Sanitary Engineering	55, 56, 87, 91, 96, 106, 111, 118, 130, 133, 134, 135, 139, 141, 149, 153, 166, 167, 175, 176, 180, 187 (Discussion: D-97, D-99, D-102, D-112, D-117, D-135)
Soil Mechanics and Foundations	43, 44, 48, 94, 102, 103, 106, 108, 109, 115, 130, 152, 155, 157, 166, 177 (Discussion: D-86, D-103, D-108, D-109, D-115)
Structural	145, 146, 147, 150, 155, 157, 158, 160, 161, 162, 163, 164, 165, 166, 168, 170, 175, 177, 179, 181, 182, 183, 185, 188 (Discussion: D-51, D-53, D-54, D-59, D-61, D-66, D-72, D-77, D-100, D-101, D-103, D-109, D-121, D-125 D-126, D-127, D-128, D-136)
Surveying and Mapping	50, 52, 55, 60, 63, 65, 68, 121, 138, 151, 152, 172, 173 (Discussion: D-60, D-65)
Waterways	123, 130, 135, 148, 154, 159, 165, 166, 167, 169, 181 (Discussion: D-19, D-27, D-28, D-56, D-70. D-71, D-78, D-79, D-80, D-112, D-113, D-115, D-123, D-135)

A constant effort is made to supply technical material to Society members, over the entire range of possible interest. Insofar as your specialty may be covered inadequately in the foregoing list, this fact is a gage of the need for your help toward improvement. Those who are planning papers for submission to "Proceedings-Separates" will expedite Division and Committee action measurably by first studying the ASCE "Guide for Development of Proceedings-Separates" as to style, content, and format. For a copy of this pamphlet, address the Manager, Technical Publications, ASCE, 33 W. 39th Street, New York 18, N. Y.

The Society is not responsible for any statement made or opinion expressed in its publications

Published at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York 18, N. Y. Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

Proceedings—Vol. 79 April, 1953 Separate No. 183

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

GROUP LOADINGS APPLIED TO THE ANALYSIS OF FRAMES

By I. F. Morrison[1]

Synopsis

In the analysis of highly indeterminate frame structures, the task of setting up and solving the elastic equations becomes time-consuming. Several methods have been developed by which the work may be shortened considerably. Of these, among the classical methods, perhaps that of group loading is the best and most comprehensive method.

The paper sets forth, in detail, the theory of this method in analytical form. Examples in algebraic form, followed by illustrative numerical computations, are used to make clear the engineering application of the theory.

A still further reduction in the number of elastic equations is effected by the choice of a statically indeterminate primary structure. A numerical application of this is also included.

Introduction

The primary purpose of the theory of group loadings is a reduction in the effort required to solve the simultaneous linear equations that arise in the analysis of statically indeterminate structures.

The customary procedure for such analysis is to select a statically determinate primary structure by the choice of such redundant forces and couples as are necessary to obtain the complete solution. This is done by making cuts through the members, inserting hinges, and removing reaction components and other forces in the original structure. The number of these is always n, for an n-fold indeterminate case. By any one of several well-known procedures, a set of simultaneous equations results:

$$\sum \delta_{ij} X_j = \delta_{io} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Note.—Written comments are invited for publication; the last discussion should be submitted by October 1, 1953.

[1] Prof. of Applied Mech., Dept. of Civ. and Municipal Eng., Univ. of Alberta, Edmonton, Alt., Canada.

in which the terms X_j are the redundants. The subscripts j and i are integers from 1 to n. The terms δ_{ij} are coefficients depending only on the size and shape of the original structure and the terms δ_{io} are the loading terms that depend both on geometry and on the load and its arrangement. The required stresses are then obtained by the principle of superposition. By the general term "stresses" in this paper is meant reactions, moments, direct stresses in truss members, and other forces of this kind. The superposition of stresses yields

$$Z = Z_0 - z_1 X_1 - z_2 X_2 - z_3 X_3 \ldots - z_n X_n \ldots\ldots\ldots\ldots (2)$$

in which the terms z_i are the stresses produced by the loading $X_i = -1$ and the terms Z_0 are the stresses produced by the actual loading on the statically determinate primary structure.

In the procedure to be followed in this paper, the elements in the square matrix, δ_{ij}, are found from the self-stressed conditions obtained by placing in

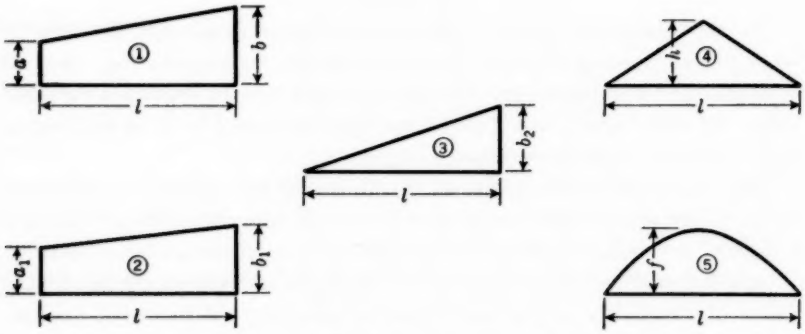


FIG. 1.—PARTICULAR CASES IN THE EVALUATION OF INTEGRALS

turn each X_j-term equal to -1, whereas all the other redundants are assumed to be zero. Each of these conditions is called "Condition $X_j = -1$." For frames or girders, sketches are made of each of the bending moment diagrams for the statically determinate primary structure with the single loadings $X_j = -1$ and from them are obtained the elements of the structure matrix by means of the Maxwell-Mohr work equation:

$$\delta_{ij} = \Sigma \int_0^l \frac{N_i N_j \, ds}{E A} + \Sigma \int_0^l \frac{m_i m_j \, ds}{E I} + \cdots\ldots\ldots\ldots (3)$$

in which E denotes the modulus of elasticity, A is the cross-sectional area of a member, N is the direct stress in a member, m is the moment produced by a unit load, s represents distance, and I is the moment of inertia. Eq. 3 is not the complete work equation. The terms relate to frames. The first term is usually small relative to the second and, therefore, only the last term will be retained as is customary. Other terms dependent on temperature changes,

shrinkage, and displacement of supports have not been included. Neglecting the first term, $\delta_{ij} = \sum \int_0^l \frac{m_i\, m_j\, ds}{E\, I}$ represents the displacements, in an assigned direction at the point of application i of the redundant X_i, produced by the loading $X_j = -1$—that is, the work done by $X_i = -1$ as a result of the deformation produced by $X_j = -1$.

The summation incorporates all the members of the frame. The modulus of elasticity will be assumed to be constant for the frame and I will be assumed to be constant for each member.

The definite integrals, which are in all cases quite simple, can best be evaluated by means of the Müller-Breslau tables,[2] a part of which follows, with the notation defined in Fig. 1. All the cases required in this paper are listed here, and case numbers are encircled in Fig. 1.

Case	$\int m_i\, m_j\, ds$
1–1	$\frac{l}{3}(a^2 + a\,b + b^2)$
1–2	$\frac{l}{6}[a_1(2\,a + b) + b_1(a + 2\,b)]$
1–3	$l\,\frac{b_2}{6}(2\,a + b)$
3–4	$\frac{h\,b_2\,l}{4}$
3–5	$\frac{f\,b_2\,l}{3}$

Because of the Maxwell theorem, the outlined procedure always leads to a symmetrical matrix and, therefore, it is necessary to determine $\frac{n\,(n+1)}{2}$ elements of the matrix, if n is the number of redundants. This task, in itself, may be long, but it can always be done, and the set of simultaneous linear equations then must be solved for the redundants X_j.

If any, or all, of the coefficients δ_{ij}, in which $i \neq j$, can be reduced to zero through a proper choice of the redundants, the effort of setting up and solving the equations is thereby reduced. There are several methods by which this can be accomplished. The method of group loadings is one of them.

Group Loadings

Group loadings are used to reduce the set of equations (Eq. 1) to one-term simultaneous equations of the form: $\delta_{ii}\, X_i = \delta_{io}$, the simple solution of which is $X_i = \frac{\delta_{io}}{\delta_{ii}}$, by making $\delta_{ij} \neq 0$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Such a process is called "orthogonalization" of the equations.

A necessary and sufficient choice of the redundants for this purpose is, however, not apparent except in very simple cases; and it is usually desired to

[2] "Die Graphische Statik der Baukonstruktionen," Vol. II, Pt. 2, Alfred Kröner Verlag, Leipzig, Germany, p. 56.

discover the conditions that will lead to such orthogonalization. To accomplish this, a linear relationship is established between the chosen X_i and a group of secondary redundants Y_k by means of the linear transformation:

$$X_j = \sum \alpha_{jk} Y_k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which k is an integer from 1 to n. This relationship is shown in detail in Table 1, which the writer will call the α-pattern.

The number of Y-terms and X-terms will always be n. Thus, the α-pattern will form a square matrix of the order n.

If Y_k equals -1, all other Y-values being equal to zero, a group of values, $X_j = -\alpha_{jk}$, results so that, instead of a set of self-stressed conditions caused by a single redundant loading $X_j = -1$, one has a set of self-stressed conditions caused by a group of redundant loadings expressed as $X_j = -\alpha_{jk}$, which is called condition $Y_k = -1$. Considering the terms Y_k to be the redundants provides a work equation that is similar to Eq. 1. This equation

TABLE 1.—THE α-PATTERN

X_j	Y_k				
	Y_a	Y_b	Y_c	...	Y_n
X_a	α_{aa}	α_{ab}	α_{ac}	...	α_{an}
X_b	α_{ba}	α_{bb}	α_{bc}	...	α_{bn}
X_c	α_{ca}	α_{cb}	α_{cc}	...	α_{cn}
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
X_n	α_{na}	α_{nb}	α_{nc}	...	α_{nn}

is $\sum \gamma_{ik} Y_k = \gamma_{io}$, in which the coefficients γ_{io} are the loading terms obtained from the group loading Y_i and the displacements of the structure produced by the actual loads. It is the γ_{ik}-terms that are to be orthogonalized. This process is expressed by the condition equations: $\gamma_{ik} \neq 0$ when $i = k$ and $\gamma_{ik} = 0$ when $i \neq k$. When this has been accomplished, Y_i becomes equal to $\frac{\gamma_{io}}{\gamma_{ii}}$, and the redundants X_j are obtained by substitution in Eqs. 4, with k equal to i. Thus, the α-pattern is to be so formed that the following condition prevails:

$$\sum \int \frac{m_k \, m_i \, ds}{E \, I} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which k is not equal to i. The moments m_k and m_i are produced by the group loadings, $Y_k = -1$ and $Y_i = -1$. Eq. 5 makes the equilibrium condition, $Y_k = -1$, orthogonal to the displacement condition of $Y_i = -1$.

It is useful to let $\bar{m}_i$ be the moment produced by the loads $X_i = -1$, and to let m_i be the moment produced by the loads $Y_i = -1$. For abbreviation, the following will be used:

$$\delta_{ik} = \sum \int \frac{\bar{m}_i \, \bar{m}_k \, ds}{E \, I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

which is the work done by the redundant, $X_k = -1$, with the displacement produced by the load, $X_i = -1$. Obviously, $\delta_{ik} = \delta_{ki}$. Similarly,

$$\beta_{ik} = \sum \int \frac{\bar{m}_i \, m_k \, ds}{E \, I} \quad \ldots\ldots (6b)$$

is the work done by the load condition, $Y_k = -1$, with the displacement produced by the load, $X_i = -1$ and $\beta_{ik} \neq \beta_{ki}$. The work accomplished by the group loading, $Y_k = -1$, with the displacement produced by the load, $X_i = -1$, is not equal to the work done by $Y_i = -1$ with the displacement $X_k = -1$. Finally,

$$\gamma_{ik} = \sum \int \frac{m_i \, m_k \, ds}{E \, I} \quad \ldots\ldots (6c)$$

is the work done by the load condition, $Y_k = -1$, with the displacements, $Y_i = -1$, and $\gamma_{ik} = \gamma_{ki}$.

Setting $m_j = \sum \alpha_{ij} \, \bar{m}_i$, and omitting the summation signs that incorporate the members, the second member of Eq. 6c can be written as

$$\int \frac{m_j \, m_k \, ds}{E \, I} = \sum \int \alpha_{ij} \frac{\bar{m}_i \, m_k \, ds}{E \, I} = \sum \alpha_{ij} \int \frac{\bar{m}_i \, m_k \, ds}{E \, I} \quad \ldots\ldots (7)$$

in which the summation is to be taken for the load group. Thus,

$$\gamma_{jk} = \sum \alpha_{ij} \, \beta_{ik} \quad \ldots\ldots (8)$$

In order to obtain a system of orthogonal functions, α_{ij}, γ_{jk} must be placed equal to zero, with j not equal to k, yielding—

$$\sum \alpha_{ij} \, \beta_{ik} = 0, \qquad j \neq k$$

from which it is desired to determine the coefficient α_{ij}. One way to do this will be described herein. For the group Y_a, it is suggested to choose arbitrarily the single value, $\alpha_{aa} = +1$, and to make all other values of α_{ia}, in the vertical column under Y_a in Table 1, zero. Thus, $X_a = Y_a$ and, therefore, condition $Y_a = -1$ becomes equivalent to condition $X_a = -1$.

For the group Y_b,

$$\gamma_{ba} = \alpha_{ab} \, \beta_{aa} + \alpha_{bb} \, \beta_{ba} + \alpha_{cb} \, \beta_{ca} + \ldots = 0 \quad \ldots\ldots (9a)$$

from which it is desired to determine α_{ab}. To do this, it is assumed that $\alpha_{bb} = +1$, and all other values of α_{ib} equal zero excepting α_{ab}. Then,

$$\alpha_{ab} = -\frac{\beta_{ba}}{\beta_{aa}} \quad \ldots\ldots (9b)$$

and

$$\gamma_{ab} = \alpha_{aa} \, \beta_{ab} + \alpha_{ba} \, \beta_{bb} + \alpha_{ca} \, \beta_{cb} + \ldots = 0 \quad \ldots\ldots (9c)$$

Then, because all α_{ia}-values have been assumed to be zero except α_{aa},

$$\beta_{ab} = 0 \quad \ldots\ldots (9d)$$

For the group Y_c, there are two equations:

$$\left.\begin{aligned}\gamma_{cb} &= \alpha_{ac}\,\beta_{ab} + \alpha_{bc}\,\beta_{bb} + \alpha_{cc}\,\beta_{cb} + \ldots = 0\\ \gamma_{ca} &= \alpha_{ac}\,\beta_{aa} + \alpha_{bc}\,\beta_{ba} + \alpha_{cc}\,\beta_{ca} + \ldots = 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (10)$$

If α_{cc} is set equal to + 1, Eqs. 10 yield

$$\alpha_{bc} = -\frac{\beta_{cb}}{\beta_{bb}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11a)$$

and

$$\alpha_{ac} = -\frac{\alpha_{bc}\,\beta_{ba}}{\beta_{aa}} - \frac{1\,\beta_{ca}}{\beta_{aa}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11b)$$

Also, the conditions:

$$\gamma_{ac} = 0 \text{ and } \gamma_{bc} = 0$$

yield

$$\beta_{ac} = 0 \text{ and } \beta_{bc} = 0.$$

This process can be continued until all the required α_{ij}-values have been determined. The α-pattern of Table 1 becomes the new arrangement shown in Table 2.

TABLE 2.—THE α-PATTERN CONFORMING TO EQ. 5

X_i	Y_k					
	Y_a	Y_b	Y_c	Y_d	...	Y_n
X_a	+1	α_{ab}	α_{ac}	α_{ad}	...	α_{an}
X_b	0	+1	α_{bc}	α_{bd}	...	α_{bn}
X_c	0	0	+1	α_{cd}	...	α_{cn}
X_d	0	0	0	+1	...	α_{dn}
.	.	.	.	.	...	.
.	.	.	.	.	...	.
X_n	0	0	0	0	...	+1

This is not the only arrangement of the arbitrarily chosen values of α. Others are possible and sometimes more desirable. For example, when the structure is symmetrical, it is expedient to take advantage of that fact in choosing the α-values. However, the number of the α_{ij}-values that can be chosen arbitrarily is always $\frac{n\,(n+1)}{n}$.

The β-values may be computed systematically. The term δ_{ia} is the displacement corresponding to the redundant $X_i = -1$ produced by $X_a = -1$ but, because X_a is equal to Y_a,

$$\beta_{ia} = \delta_{ia} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

All values of β_{ib} are the displacements corresponding to the redundants $X_i = -1$ of the primary structure produced by the loads $Y_b = -1$; that is, by the group loadings, $X_b = -1$ and $X_a = -\alpha_{ab}$.

Thus, putting $m_b = \bar{m}_b + \alpha_{ab}\,\bar{m}_a$ into Eq. 6*b*, with k equal to b, yields

$$\beta_{ib} = \delta_{ib} + \alpha_{ab}\,\delta_{ia} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12b)$$

and, because $\alpha_{ab} = -\frac{\beta_{bc}}{\beta_{aa}} = -\frac{\delta_{ba}}{\delta_{aa}}$ from Eq. 9*b*,

$$\beta_{ib} = \delta_{ib} - \frac{\delta_{ba}\,\delta_{ia}}{\delta_{aa}} \qquad (12c)$$

for all values of i. In the same way,

$$\beta_{ic} = \delta_{ic} - \frac{\delta_{ca}\,\delta_{ia}}{\delta_{aa}} - \frac{\beta_{cb}\,\beta_{ib}}{\beta_{bb}} \qquad (12d)$$

Examples

A careful study of the following examples will make the process clearer. After an α-pattern has been established for a certain type of structure, it can be retained and the numerical values filled in.

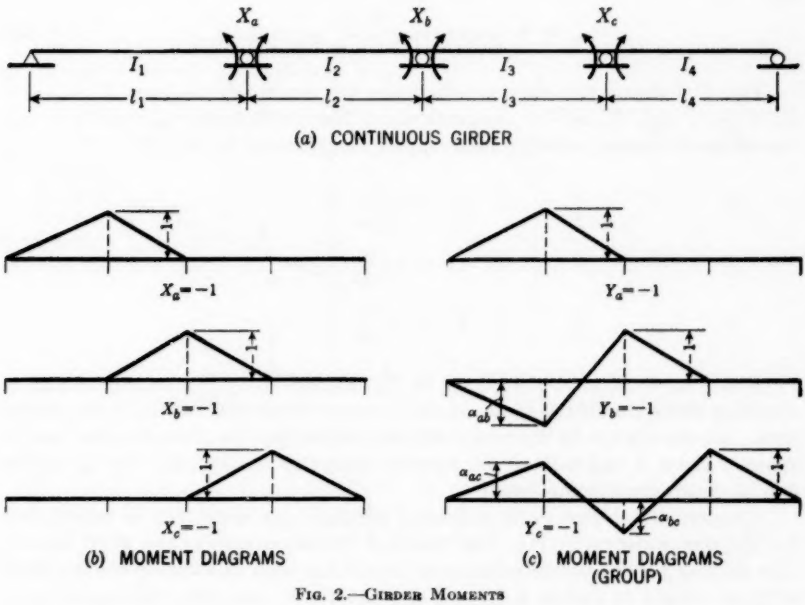


Fig. 2.—Girder Moments

Continuous Girder.—Fig. 2(*a*) shows a girder on five supports. The length of the structure is l, and the value $l' = \frac{l}{l}$—called the modified span—will be used for abbreviation. The redundants have been taken as X_a, X_b, and X_c, which are the moments at the inner supports. Fig. 2(*b*) shows the moment diagrams for the conditions $X_a = -1$, $X_b = -1$, and $X_c = -1$, respectively. By means of the Müller-Breslau tables, or by direct integration, the following

can be written:

$$\delta_{aa} = \frac{l'_1 + l'_2}{3}, \qquad \delta_{ab} = \frac{l'_2}{6}, \qquad \delta_{ac} = 0$$

$$\delta_{ba} = \frac{l'_2}{6}, \qquad \delta_{bb} = \frac{l'_2 + l'_3}{3}, \qquad \delta_{bc} = \frac{l'_3}{6}$$

$$\delta_{ca} = 0, \qquad \delta_{cb} = \frac{l'_3}{6}, \qquad \delta_{cc} = \frac{l'_3 + l'_4}{3}$$

By Eqs. 9*b* and 12*a*,

$$\alpha_{ab} = -\frac{\delta_{ba}}{\delta_{aa}} = -\frac{l'_2}{2\,(l'_1 + l'_2)} \qquad (13a)$$

Eqs. 11 and 12 lead to

$$\alpha_{bc} = -\frac{2\,l'_3\,(l'_1 + l'_2)}{4\,(l'_2 + l'_3)\,(l'_1 + l'_2) - (l'_2)^2} \qquad (13b)$$

and

$$\alpha_{ac} = +\frac{l'_3\,l'_2}{4\,(l'_2 + l'_3)\,(l'_1 + l'_2) - (l'_2)^2} \qquad (13c)$$

Fig. 1(*c*) shows the moment diagrams for the group loadings $Y_a = -1$, $Y_b = -1$, and $Y_c = -1$, respectively. The coefficients γ_{ab}, γ_{bc}, and γ_{ca} are all equal to zero, and the following expressions can be solved:

$$\left.\begin{aligned} Y_a &= +\frac{\gamma_{ao}}{\gamma_{aa}} \\ Y_b &= +\frac{\gamma_{bo}}{\gamma_{bb}} \\ Y_c &= +\frac{\gamma_{co}}{\gamma_{cc}} \end{aligned}\right\} \qquad (14)$$

From Eq. 4 values are obtained for X_a, X_b, and X_c. The moment diagram resulting from the actual loads on the primary structure has not been shown here. It can always be drawn for any loads that may be placed on the simple spans. From it and each of the moment diagrams of Fig. 1(*c*), the γ_{io}-values can be determined immediately.

Unsymmetrical Bent.—As a second example, an α-pattern is established for the frame shown in Fig. 3(*a*) which is indeterminate to the third degree. The statically determinate primary structure has been selected by the insertion of three hinges at points a, b, and c, respectively, and the redundants have been taken as the moments at those points. The location of the hinge at point a had been left indeterminate for convenience. The procedure involves making the substitution, $\frac{l_c}{l_b} = \nu$, and finding $h_0 = \frac{2}{3}\frac{(h_1\,\nu + h_2)}{(1 + \nu)}$. Instead of following the previous procedure, an adequate α-pattern is adopted through the choice of six of the α-values as shown in Fig. 3(*b*).

Fig. 4(*a*) shows the bending moment diagrams for the following conditions: $X_a = -1$, $X_b = -1$, and $X_c = -1$. The symbol m in Fig. 4 designates mo-

ments at the corner joints and rigid column bases. These moments are

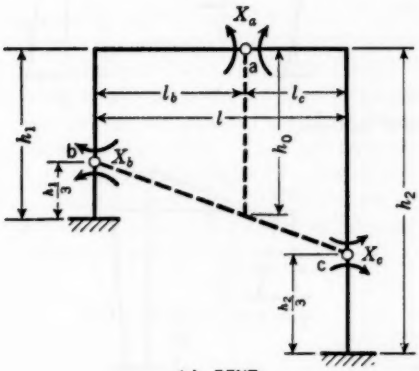
$$m_1 = \frac{h_1(\nu+1)}{(h_1\nu+h_2)}, \quad m_3 = \frac{h_2}{(h_1\nu+h_2)}, \quad m_5 = \frac{h_1}{(h_1\nu+h_2)}$$

$$m_2 = \frac{h_2(\nu+1)}{(h_1\nu+h_2)}, \quad m_4 = \frac{h_2\nu}{(h_1\nu+h_2)}, \quad m_6 = \frac{h_1\nu}{(h_1\nu+h_2)}$$

Therefore, $\nu m_3 = m_4$, and $\nu m_5 = m_6$. Because $Y_a = -1$ is the same as $X_a = -1$, Eq. 6c yields the following by means of the Müller-Breslau tables:

$$\gamma_{aa} = \frac{4}{9h^2_0}\left[\frac{h'_1 h^2_1}{4} + \frac{h'_2 h^2_2}{4} + \frac{l'}{3}(h^2_1 + h_1 h_2 + h^2_2)\right] \ldots\ldots (15)$$

in which h' is equal to $\frac{h}{I}$. Because β_{aa} is not equal to zero, if it is desired to make α_{ab} equal to zero, it is necessary that (from Eq. 9a) $\alpha_{bb}\beta_{ba} + 1\,\beta_{ca} = 0$



(a) BENT

X_j	Y_k		
	Y_a	Y_b	Y_c
X_a	+1	0	α_{ac}
X_b	0	α_{bb}	$-\alpha_{bc}$
X_c	0	+1	+1

(b) THE α-PATTERN

FIG. 3.—RIGID FRAME ANALYSIS

so that

$$\alpha_{bb} = -\frac{\beta_{ca}}{\beta_{ba}} = -\frac{\delta_{ca}}{\delta_{ba}} \ldots\ldots (16a)$$

This relationship yields

$$\alpha_{bb} = +\frac{h_1}{h_2} \ldots\ldots (16b)$$

The condition $\alpha_{bb} = +1$ could have been chosen, but it would have been necessary to compute α_{ab} which would, in general, not be zero.

The column under Y_b in Fig. 3(a) has been established, and Fig. 4(b) shows the condition, $Y_b = -1$. From Fig. 4(b), and by means of the Müller-Breslau tables,

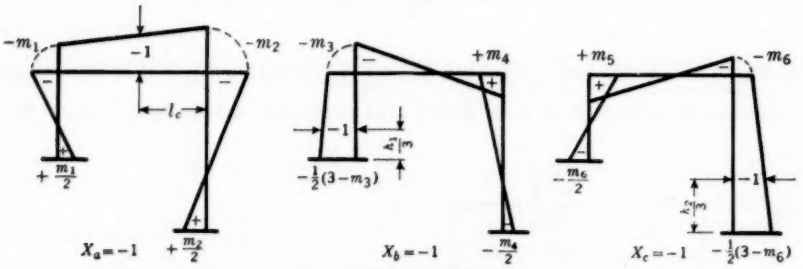
$$\gamma_{bb} = \frac{3}{4}\left[\frac{h'_1 h^2_1 + h'_2 h^2_2}{h^2_2}\right] \ldots\ldots (17)$$

The next step is to obtain expressions for α_{ac} and α_{bc}. From Eq. 11*a*, the following is true:

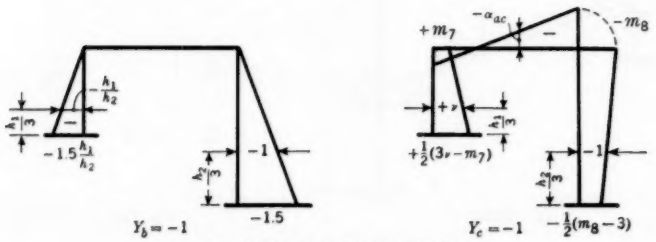
$$-\alpha_{bc} = -\frac{\beta_{cb}}{\beta_{bb}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

From the conditions: $X_c = -1$, $Y_b = -1$, and $X_b = -1$, $Y_b = -1$, it is found that, according to Eq. 6*b*, $\beta_{cb} = +\frac{3}{4}h'_2$, $\beta_{bb} = +\frac{3}{4}\frac{h'_1 h_2}{h_2}$, so that

$$\alpha_{bc} = +\frac{h'_2 h_2}{h'_1 h_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18b)$$



(*a*) MOMENT DIAGRAMS



(*b*) MOMENT DIAGRAMS (GROUP)

FIG. 4.—MOMENTS IN THE FRAME OF FIG. 3

If ν is so chosen that it equals $\frac{h'_1 h_1}{h'_2 h_2}$, it follows that $\alpha_{bc} = \frac{1}{\nu}$. To compute α_{ac}, substitution into Eq. 11*b* yields

$$\alpha_{ac} = \frac{1}{\nu}\frac{\beta_{ba}}{\beta_{aa}} - \frac{1}{\beta_{aa}}\frac{\beta_{ca}}{\beta_{cc}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

Because $\delta_{ca} = -\frac{h_1}{h_2}\delta_{ba}$, it may be written that $\alpha_{ac} = +\frac{\delta_{ba}}{\delta_{aa}}\frac{(h_2 + \nu h_1)}{\nu h_2}$, which is equivalent to

$$\alpha_{ac} = \frac{3}{2}l' h_0 \frac{\left[\frac{1}{\nu}(2h_1 + h_2) - (2h_2 + h_1)\right]}{\frac{3}{2}(h'_1 h^2_1 + h'_2 h^2_2) + 2l'(h^2_1 + h_1 h_2 + h^2_2)} \quad \ldots\ldots (19b)$$

Fig. 4(*b*) shows the moment diagram for the condition, $Y_c = -1$, from which the coefficient γ_{cc} may be determined.

Because the purpose of this analysis is to demonstrate the method rather than to obtain algebraic formulas for any particular frame, the algebra will not be carried further. In practice, one works directly with the numerical values. Moreover, in this particular example, the elastic center method may perhaps be shorter but it has the same purpose—namely, to reduce the elastic equations to one-term equations.

TABLE 3.—THE α-PATTERN (SYMMETRIC FRAME)

X_j \ Y_k	Y_a	Y_b	Y_c
X_a	+1	0	0
X_b	0	+1	−1
X_c	0	+1	+1

If the frame is symmetrical, the α-pattern becomes quite simple, because $\nu = 1$ and $\frac{h_1}{h_2} = 1$. Also, $\alpha_{ac} = 0$ and $\alpha_{bc} = +1$. Therefore, the α-pattern is as shown in Table 3.

INDETERMINATE PRIMARY STRUCTURES

In the preceding analysis, the primary structure in each case has been chosen as a statically determinate structure. The only reason for this is that the bending moment diagrams always can be readily sketched.

However, if there are at hand indeterminate cases that have previously been solved, the primary structure may be chosen as being made up of statically indeterminate parts. In a previous paper, the writer has directed attention to the "Rahmenformeln" of A. Kleinlogel, which serves the purpose admirably.[3]

Rigid Frame.—A simple numerical example is presented to show the application of the theory of group loadings, assisted by the use of an n'-fold statically indeterminate primary structure.

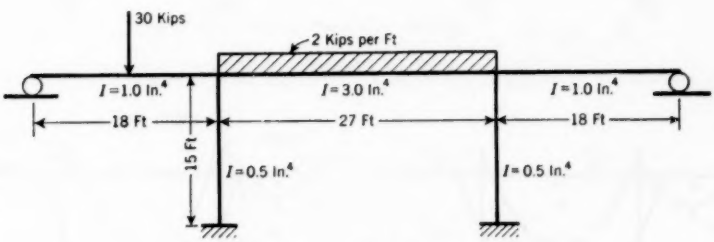


FIG. 5.—LOADED STRUCTURE

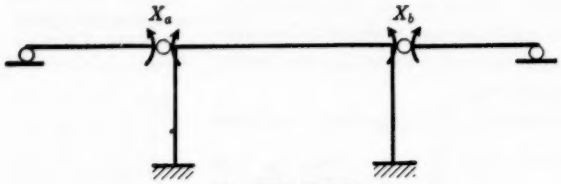
Fig. 5 shows the original structure. It is symmetrical, and it is 5-fold statically indeterminate. The primary structure can be selected in several ways, one of which appears in Fig. 6(*a*). It is 3′-fold statically indeterminate, and the number of redundants is two.

Because the original structure is symmetrical, the two redundants X_a and X_b have been symmetrically arranged. Therefore, the α-pattern for a sym-

[3] "On the Extended Use of Kleinlogel's Rahmenformeln," by I. F. Morrison, *Canadian Journal of Research*, F, 26, December, 1948, pp. 552–564.

metrical 2-fold statically indeterminate structure may be chosen as shown in Fig. 6(*b*).

Fig. 7 shows the two cases, $Y_a = -1$, $Y_b = -1$, and also "Condition 0," defining the stresses Z_0 (see Eq. 2) produced by the loads on the primary structure.



(*a*) REDUNDANTS

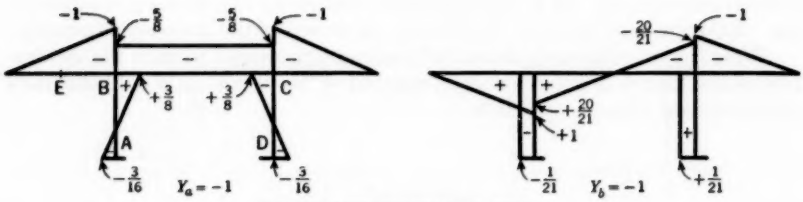
X_j	Y_k	
	Y_a	Y_b
X_a	+1	−1
X_b	+1	+1

(*b*) THE a-PATTERN

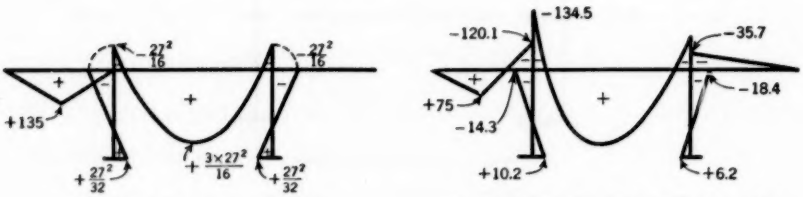
FIG. 6.—PRIMARY STRUCTURE

The γ-values can be computed directly from the moment diagrams of Fig. 7 with the aid of the Müller-Breslau tables.

$$\gamma_{aa} = +16.57, \qquad \gamma_{ao} = -1{,}290.9$$
$$\gamma_{cc} = +14.86, \qquad \gamma_{bo} = +\ 607.5.$$



(*a*) MOMENT DIAGRAMS (GROUP)



(*b*) CONDITION 0, IN KIP-FEET

(*c*) COMPLETE MOMENTS, IN KIP-FEET

FIG. 7.—MOMENTS IN THE FRAME OF FIG. 5

The group loadings may be computed as $Y_a = -\dfrac{1{,}290.9}{16.57} = -77.9$; and $Y_b = +\dfrac{607.5}{14.86} = +42.2$. The redundants are $X_a = -77.9 - 42.2 = -120.1$; and $X_b = -77.9 + 42.2 = -35.7$.

Table 4, based on Eq. 2, shows the moments at several points. From these values the complete bending moment diagram can be drawn as shown in Fig. 7(*c*).

Three-Bay Frame.—As a second numerical example, the frame shown in Fig. 8(*a*) is considered. This structure is 9-fold statically indeterminate.

TABLE 4.—MOMENT VALUES, IN KIP-FEET, FOR THE STRUCTURE OF FIG. 5

Condition	Term	Point (Fig. 7 (*a*))								
		A	B	B (BC)	B (BA)	C (CD)	C (CB)	C	D	E
0	Z_0	+22.78	0	− 45.56	−45.56	−45.56	−45.56	0	+22.78	+135.0
$Y_a = -1$	$-Y_a z_1$	−14.61	− 77.9	− 48.70	+29.22	+29.22	−48.70	−77.9	−14.61	− 38.95
$Y_b = -1$	$-Y_b z_2$	+ 2.01	− 42.2	− 40.19	+ 2.01	− 2.01	+40.19	+42.2	− 2.01	− 21.10
Final	Z	+10.18	−120.1	−134.45	−14.33	−18.35	−54.07	−35.7	+ 6.16	+ 74.95

The 6′-fold statically indeterminate primary structure is shown in Fig. 8(*b*), leaving three redundant quantities; and the α-pattern is shown subsequently in Fig. 9(*d*). In this α-pattern, $V_0 = X_a$, $H_0 = X_b$, and $M_0 = X_c$.

Fig. 9(*a*) shows the conditions, $X_a = -1$, $X_b = -1$, and $X_c = -1$. Because of the antisymmetry of the condition, $X_a = -1$, the following

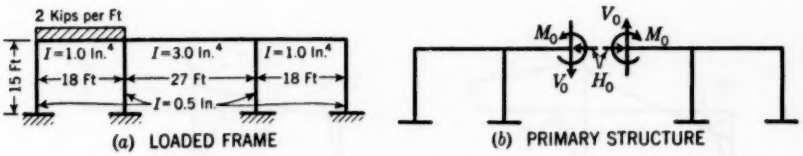


FIG. 8.—THREE-BAY FRAME

equations result: $\delta_{ca} = 0$ and $\delta_{ba} = 0$. Therefore, if $\alpha_{ab} = 0$, it follows that $\alpha_{bb} = \frac{0}{0} = 1$.

From Eqs. 11*b* and 12*a*, $\alpha_{ac} = \frac{\alpha_{bc}\,\delta_{ba} - \delta_{ca}}{\delta_{aa}}$, and because $\delta_{ca} = \delta_{ba} = 0$, it is true that $\alpha_{ac} = 0$. Also, from Eq. 11*a*, $\alpha_{bc} = \frac{\beta_{cb}}{\beta_{bb}}$. However,

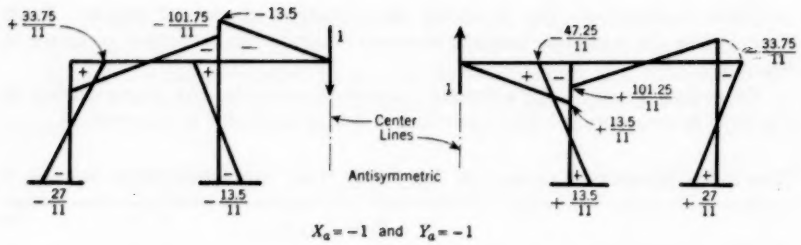
$$\beta_{cb} = \delta_{cb} - \frac{\delta_{ba}\,\delta_{ca}}{\delta_{aa}} = \delta_{cb}$$

and

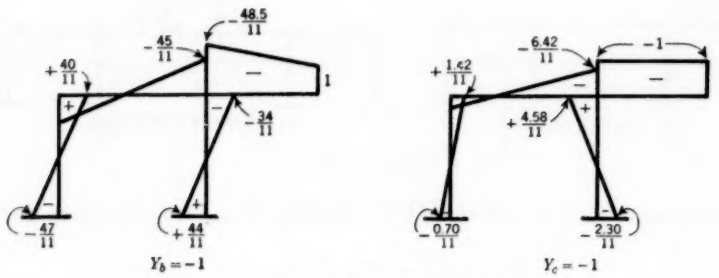
$$\beta_{bb} = \delta_{bb} - \frac{\delta_{ba}\,\delta_{ba}}{\delta_{aa}} = \delta_{bb}$$

and, therefore, $\alpha_{bc} = \frac{\delta_{cb}}{\delta_{bb}}$.

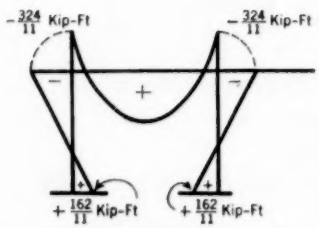
From Fig. 9(*a*), $\frac{1}{2}\,\delta_{cb} = \frac{1{,}247.5}{121}$, $\frac{1}{2}\,\delta_{bb} = \frac{43{,}312.5}{121}$, and $\alpha_{bc} = +\,0.02882$.



(*a*) MOMENT DIAGRAMS



(*b*) MOMENT DIAGRAMS (GROUP)



(*c*) CONDITION 0

X_j	Y_k		
	Y_a	Y_b	Y_c
X_a	+1	0	α_{ac}
X_b	0	α_{bb}	$-\alpha_{bc}$
X_c	0	+1	+1

(*d*) THE α-PATTERN

FIG. 9.—MOMENTS IN THE FRAME OF FIG. 8

The α-pattern in Fig. 9(d) is thus complete. Fig. 9(b) shows the conditions, $Y_b = -1$ and $Y_c = -1$, and Fig. 9(c) shows Condition 0 readily obtained from the Kleinlogel "Rahmenformeln." The γ-values are as follows:

$$\gamma_{aa} = \frac{216{,}512.7}{121}, \qquad \gamma_{ao} = -\frac{360{,}855}{121}$$

$$\gamma_{bb} = \frac{162{,}602.5}{121}, \qquad \gamma_{bo} = -\frac{26{,}730}{121}$$

$$\gamma_{cc} = \frac{1{,}843.2}{121}, \qquad \gamma_{co} = -\frac{26{,}730}{121}.$$

The group loadings are: $Y_a = -\frac{360{,}855}{216{,}513} = -1.67$ kips; $Y_b = -\frac{26{,}730}{162{,}602} = -0.164$ kips; and $Y_c = -\frac{26{,}730}{1{,}843.2} = -14.5$ kip-ft. The values of the redundants are

$$V_0 = -1.67 \text{ kips}$$
$$H_0 = +0.254 \text{ kips}$$
$$M_0 = -14.66 \text{ kip-ft.}$$

It seems unnecessary to carry the numerical computation further because the procedure has been shown in the preceding numerical example.

CURRENT PAPERS

Proceedings-Separate Number	Title and Author	Discussion closes*
159	"Development of a Flood-Control Plan for Houston, Tex.," by Ellsworth I. Davis	June 1
160	"Ice Pressure Against Dams: Studies of the Effects of Temperature Variations," by Bertil Löfquist	June 1
161	"Ice Pressure Against Dams: Some Investigations in Canada," by A. D. Hogg	June 1
162	"Ice Pressure Against Dams: Experimental Investigations by the Bureau of Reclamation," by G. E. Monfore	June 1
163	"A Comparison of Design Methods for Airfield Pavements," Progress Report of the Committee on Correlation of Runway Design Procedures of the Air Transport Division	June 1
164	"Water Supply Engineering," Report of Committee on Water Supply Engineering of the Sanitary Engineering Division for the Period Ending September 30, 1951	July 1
165	"Design Curves for Anchored Steel Sheet Piling," by Walter C. Boyer and Henry M. Lummis, III	July 1
166	"The Design of Flexible Bulkheads," by James R. Ayers and R. C. Stokes	July 1
167	"Sewage Disposal in Tidal Estuaries," by Alexander N. Diachishin, Seth G. Hess, and William T. Ingram	July 1
168	"Special Design Features of the Yorktown Bridge," Maurice N. Quade	July 1
169	"Rating Curves for Flow over Drum Gates," by Joseph N. Bradley	Aug. 1
170	"Rapid Computation of Flexural Constants," by Thomas G. Morrison	Aug. 1
171	"Unified Mass-Transportation System for New York," by William Reid	Aug. 1
172	"Aeronautical Charting and Mapping," by Charles A. Schanck	Aug. 1
173	"Electronic Devices for Air Transport," by F. B. Lee	Aug. 1
174	"Zoning Maps for Airports," by Benjamin Everett Beavin, Sr.	Aug. 1
175	"Design of Side Walls in Chutes and Spillways," by D. B. Gumensky	Aug. 1
176	"Advances in Sewage Treatment and Present Status of the Art," Progress Report of the Committee of the Sanitary Engineering Division on Sewerage and Sewage Treatment	Sept. 1
177	"Earthquake Stresses in Shear Buildings," by M. G. Salvadori	Sept. 1
178	"Rainfall Studies Using Rain-Gage Networks and Radar," by H. E. Hudson, Jr., G. E. Stout, and F. A. Huff	Sept. 1
179	"Stiffness Charts for Gusseted Members Under Axial Load," by John E. Goldberg	Sept. 1
180	"A Direct Step Method for Computing Water-Surface Profiles," by Arthur A. Ezra	Sept. 1
181	"Slackwater Improvement of the Columbia River," by O. E. Walsh	Oct. 1
182	"Hipped Plate Analysis, Considering Joint Displacement," by Ibrahim Gaafar	Oct. 1
183	"Group Loadings Applied to the Analysis of Frames, by I. F. Morrison	Oct. 1
184	"Dam Modifications Checked by Hydraulic Models," by E. S. Harrison and Carl E. Kindsvater	Oct. 1
185	"Nonelastic Behavior of Bridges Under Impulsive Loads," by S. J. Fraenkel and L. E. Grinter	Oct. 1
186	"Settling Rates of Suspensions in Solids Contact Units," by A. A. Kalinske	Oct. 1
187	"The Equivalent Rectangle in Prestressed Concrete Design," by John J. Peebles	Oct. 1
188	"Laminar to Turbulent Flow in a Wide Open Channel," by W. M. Owen	Oct. 1

* Readers are urged to submit discussion applying to current papers. Forty free Separates per year are allotted to members. Mail the coupon order form found in the current issue of *Civil Engineering*.

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1953

PRESIDENT

WALTER LEROY HUBER

VICE-PRESIDENTS

Term expires October, 1953:

GEORGE W. BURPEE
A. M. RAWN

Term expires October, 1954:

EDMUND FRIEDMAN
G. BROOKS EARNEST

DIRECTORS

Term expires October, 1953:

KIRBY SMITH
FRANCIS S. FRIEL
WALLACE L. CHADWICK
NORMAN R. MOORE
BURTON G. DWYRE
LOUIS R. HOWSON

Term expires October, 1954:

WALTER D. BINGER
FRANK A. MARSTON
GEORGE W. McALPIN
JAMES A. HIGGS
I. C. STEELE
WARREN W. PARKS

Term expires October, 1955:

CHARLES B. MOLINEAUX
MERCEL J. SHELTON
A. A. K. BOOTH
CARL G. PAULSEN
LLOYD D. KNAPP
GLENN W. HOLCOMB
FRANCIS M. DAWSON

PAST-PRESIDENTS

Members of the Board

GAIL A. HATHAWAY
CARLTON S. PROCTOR

TREASURER

CHARLES E. TROUT

EXECUTIVE SECRETARY

WILLIAM N. CAREY

ASSISTANT TREASURER

GEORGE W. BURPEE

ASSISTANT SECRETARY

E. L. CHANDLER

PROCEEDINGS OF THE SOCIETY

HAROLD T. LARSEN
Manager of Technical Publications

DEFOREST A. MATTESON, JR.
Assoc. Editor of Technical Publications

PAUL A. PARISI
Asst. Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

LOUIS R. HOWSON

FRANCIS S. FRIEL
I. C. STEELE
GLENN W. HOLCOMB
FRANK A. MARSTON
NORMAN R. MOORE